1. (2.3.1) Show that the matrix $M$ is invertible and find its inverse in $M_3(\mathbb{Z})$.

$$M = \begin{pmatrix} 1 & 4 & 1 \\ 0 & 1 & -1 \\ -3 & -6 & -8 \end{pmatrix}$$

This one is just a direct computation, you can even get Mathematica to do it for you.

$$M^{-1} = \begin{pmatrix} -14 & 26 & -5 \\ 3 & -5 & 1 \\ 3 & -6 & 1 \end{pmatrix}$$

2. (2.3.2) Prove that if $R$ is a commutative ring then $AB = 1$ in $M_n(R)$ if and only if $BA = 1$ in $M_n(R)$.

**Proof.** Let $A, B \in M_n(R)$ with $AB = 1$. This forces $\det(A) \det(B) = 1$ and so $\det(A)$ is invertible (because $R$ is commutative, $\det(B)$ is a two-sided inverse). Because $R$ is commutative, we can apply the Corollary on page 96 and $A$ is invertible with inverse $B$ and $BA = A^{-1}A = 1$. \[\square\]

3. (2.3.5) Prove that $\eta: \mathbb{C} \to M_n(\mathbb{R})$ given below is a monomorphism (injective homomorphism) from $\mathbb{C}$ into $M_2(\mathbb{R})$.

$$\eta(a + bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

**Proof.** We must show that $\eta$ is a injection and that $\eta$ preserves addition and multiplication and the multiplicative identity. Let $a + bi, c + di \in \mathbb{C}$ with $a, b, c, d \in \mathbb{R}$.

To show that $\eta$ is an injection, assume $\eta(a + bi) = \eta(c + di)$, then we have

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} c & d \\ -d & c \end{pmatrix}$$

So $a = c, b = d, -b = -d, a = c$ which means $a + bi = c + di$ as required. \[\nabla\]
Now to show that $\eta$ preserves addition.

$$
\eta((a + bi) + (c + di)) = \eta((a + c) + (b + d)i) = \begin{pmatrix} a + c & b + d \\ -b - d & a + c \end{pmatrix}
= \begin{pmatrix} a & b \\ -b & a \end{pmatrix} + \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \eta(a + bi) + \eta(c + di)
$$

So $\eta$ preserves addition. \hfill ∇

Now to show that $\eta$ preserves multiplication.

$$
\eta((a + bi)(c + di)) = \eta((ac - bd) + (ad + bc)i) = \begin{pmatrix} ac - bd & ad + bc \\ -ad - bc & ac - bd \end{pmatrix}
= \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \eta(a + bi)\eta(c + di)
$$

So $\eta$ preserves the product. \hfill ∇

Now to show that $\eta$ preserves the multiplicative identity.

$$
\eta(1 + 0i) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
$$

So $\eta$ preserves the identity. \hfill ∇

Therefore $\eta$ is a monomorphism of $\mathbb{C}$ into $M_2(\mathbb{R})$. \hfill □

4. (2.3.8) Let $D \subset \mathbb{R}^{n \times n}$ be the collection of diagonal matrices,

$$
D = \left\{ \sum_{i \leq i \leq n} m_i e_{ii} \middle| m_i \in \mathbb{R} \right\}
$$

(a) Let $M, N \in D$, prove that $MN = NM$.

Proof. Let $M = \sum m_{ii} e_{ii}$ and $N = \sum n_{jj} e_{jj}$, where $m_{ii}, n_{jj} \in \mathbb{R}$, then

$$
MN = \sum m_{ii} e_{ii} \sum n_{jj} e_{jj} = \sum m_{ii} n_{jj} e_{ij} = \sum m_{ii} n_{ii} e_{ii}
= \sum n_{ii} m_{ii} e_{ii} = \sum n_{ii} m_{jj} \delta_{ij} e_{ij} = \sum n_{ii} e_{ii} \sum m_{jj} e_{jj} = NM
$$

□
(b) Let $A \in \mathbb{R}^{n \times n} \setminus D$. Prove that there exists a matrix $B \in D$ with $AB \neq BA$.

Proof. Let $A \in \mathbb{R}^{n \times n} \setminus D$. Then $A = \sum a_{ij}e_{ij}$ with $a_{ij} \in \mathbb{R}$ and there exists $k \neq \ell$ with $a_{k\ell} \neq 0$. Let $B = e_{kk}$, then

$$BA = \left( \sum_{i,j} a_{ij}e_{ij} \right) e_{kk} = \sum_{i,j} a_{ij}\delta_{jk}e_{ik} = \sum_{i} a_{ik}e_{ik}$$

and

$$AB = e_{kk} \left( \sum_{i,j} a_{ij}e_{ij} \right) = \sum_{i,j} a_{ij}\delta_{ki}e_{kj} = \sum_{j} a_{kj}e_{kj}$$

From this $(BA)_{k\ell} = 0$ because $k \neq \ell$, so $e_{k\ell}$ does not appear in the sum and $(AB)_{k\ell} = a_{k\ell} \neq 0$. So $AB \neq BA$.

5. (2.3.12) Let $F$ be a field.

(a) Show that $A \in M_n(F)$ is a zero-divisor if and only if $A$ is not invertible.

Proof. If $A$ is invertible, then $A$ cannot be a zero-divisor since $AB = 0$ implies $B = A^{-1}AB = A^{-1}0 = 0$. ▽

If $A$ is not invertible, then (by Linear Algebra) there exists a non-zero column vector $X$ with $AX = 0$. Let $B$ be the matrix each of whose columns is equal to $X$, so $B$ is not the zero matrix, but

$$AB = A \begin{pmatrix} X & \cdots & X \end{pmatrix} = \begin{pmatrix} AX & \cdots & AX \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}$$

Therefore, $A$ is a zero-divisor. ▽

So the theorem holds. □

(b) Does this hold in $M_n(\mathbb{Z})$?
Example. In $M_1(\mathbb{Z})$, the matrix $(2)$ is not invertible and is not a zero-divisor because if $B = (b)$ satisfies $AB = 0$, then

$$(2b) = AB = (0)$$

forces $b = 0$ and so $B = (b) = (0)$. $\square$